

Number-conserving cellular automaton rules

Nino Boccara[†] and Henryk Fukś[‡]

[†] *Department of Physics, University of Illinois, Chicago, USA*

boccara@uic.edu

and

DRECAM/SPEC, CE Saclay, 91191 Gif-sur-Yvette Cedex, France

[‡] *The Fields Institute for Research in Mathematical Sciences,*

Toronto ON M5T 2W1, Canada

hfuks@fields.utoronto.ca and

Department of Mathematics and Statistics, University of Guelph,

Guelph, ON N1G 2W1, Canada

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Abstract. A necessary and sufficient condition for a one-dimensional q -state n -input cellular automaton rule to be number-conserving is established. Two different forms of simpler and more visual representations of these rules are given, and their flow diagrams are determined. Various examples are presented and applications to car traffic are indicated. Two nontrivial three-state three-input self-conjugate rules have been found. They can be used to model the dynamics of random walkers.

1. Introduction

A one-dimensional cellular automaton (CA) is a discrete dynamical system, which may be defined as follows. Let $s : \mathbb{Z} \times \mathbb{N} \mapsto \mathcal{Q}$ be a function that satisfies the equation

$$s(i, t + 1) = f(s(i - r_\ell, t), s(i - r_\ell + 1, t), \dots, s(i + r_r, t)), \quad (1)$$

for all $i \in \mathbb{Z}$, and all $t \in \mathbb{N}$. \mathbb{Z} is the set of all integers, \mathbb{N} the set of nonnegative integers, and \mathcal{Q} a finite set of states, usually equal to $\{0, 1, 2, \dots, q - 1\}$. $s(i, t)$ represents the *state of site i at time t* , and the mapping $f : \mathcal{Q}^{r_\ell + r_r + 1} \rightarrow \mathcal{Q}$ is

the *CA evolution rule*. The positive integers r_ℓ and r_r are, respectively, the *left* and *right radii* of the rule. In what follows, f will be referred to as an *n-input rule*, where n is the number $r_\ell + r_r + 1$ of arguments of f . Following Wolfram [1], to each rule f we assign a *rule number* $N(f)$ such that

$$N(f) = \sum_{(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n} f(x_1, x_2, \dots, x_n) q^{q^{n-1}x_1 + q^{n-2}x_2 + \dots + q^0x_n}.$$

Cellular automata (CAs) have been widely used to model complex systems in which the local character of the evolution rule plays an essential role [2, 3, 4, 5]. In this paper, we will only consider finite CAs, and replace the set \mathbb{Z} by the set \mathbb{Z}_L of integers modulo L . Any element of the set \mathcal{Q}^L will be called an *L-cycle* or a *cyclic configuration of length L*.

Recently, following Nagel and Schreckenberg [6], many authors proposed various CA models of highway traffic flow. In the simplest models of this type, \mathbb{Z}_L represents a one-lane circular highway, and \mathcal{Q} is equal to $\{0, 1\}$. According to the value of $s(i, t)$, it is said that, at time t , site i is either empty, if $s(i, t) = 0$, or occupied by a car, if $s(i, t) = 1$. In order to complete the description of the model, an evolution rule has to be defined. For the sake of simplicity, we might assume that, at each time step, all cars move to the right neighboring site if, and only if, this site is empty. It is not difficult to verify that this rule coincides with Rule 184 defined by

$$\begin{aligned} f_{184}(0, 0, 0) &= 0, & f_{184}(0, 0, 1) &= 0, & f_{184}(0, 1, 0) &= 0, & f_{184}(0, 1, 1) &= 1, \\ f_{184}(1, 0, 0) &= 1, & f_{184}(1, 0, 1) &= 1, & f_{184}(1, 1, 0) &= 0, & f_{184}(1, 1, 1) &= 1. \end{aligned}$$

This traffic rule does not allow cars to enter or exit the highway. Therefore, during the evolution, the number of cars should remain constant, that is, starting from any initial cyclic configuration of length L , and for all $t \in \mathbb{N}$, Rule 184 should satisfy the condition

$$\begin{aligned} f_{184}(s(1, t), s(2, t), s(3, t)) &+ f_{184}(s(2, t), s(3, t), s(4, t)) + \dots \\ &+ f_{184}(s(L, t), s(1, t), s(2, t)) = s(1, t) + s(2, t) + \dots + s(L, t). \end{aligned} \quad (2)$$

Since Rule 184 may also be written

$$f_{184}(x_1, x_2, x_3) = x_2 + \min\{x_1, 1 - x_2\} - \min\{x_2, 1 - x_3\}, \quad (3)$$

condition (2) is clearly verified. If we had assumed that cars, instead of moving to the right, had to move to the left, we would have found that the evolution rule would have been Rule 226. The relations

$$\begin{aligned} f_{226}(x_1, x_2, x_3) &= f_{184}(x_3, x_2, x_1) \\ f_{226}(x_1, x_2, x_3) &= 1 - f_{184}(1 - x_1, 1 - x_2, 1 - x_3), \end{aligned}$$

show that these two rules are not fundamentally distinct.

Rules 184 and 226 are the simplest nontrivial examples of number-conserving CAs. In a previous paper [7], in order to find all the deterministic car traffic rules, we determined, up to $n = 5$, all the 2-state number-conserving CA rules. This was done using a different technique than the one presented in this paper.

For $n > 3$, not all these rules did correspond to realistic car traffic rules since they allowed vehicles to move in both directions. We found then that it was more appropriate to interpret all these 2-state number-conserving CA rules as evolution operators of one-dimensional systems of distinguishable particles.

The purpose of this paper is to derive a necessary and sufficient condition for a one-dimensional q -state n -input CA rule to be number-conserving. We will then give examples of such rules and indicate some of their applications. Our result is an illustration of a general theorem on additive conserved quantities established by Hattori and Takesue [8].

2. Number-conserving rules

Definition 2.1. A one-dimensional q -state n -input CA rule f is *number-conserving* if, for all cyclic configurations of length $L \geq n$, it satisfies

$$f(x_1, x_2, \dots, x_{n-1}, x_n) + f(x_2, x_3, \dots, x_n, x_{n+1}) + \dots \\ + f(x_L, x_1, \dots, x_{n-2}, x_{n-1}) = x_1 + x_2 + \dots + x_L. \quad (4)$$

Theorem 2.1. A one-dimensional q -state n -input CA rule f is number-conserving if, and only if, for all $(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n$, it satisfies

$$f(x_1, x_2, \dots, x_n) = x_1 + \sum_{k=1}^{n-1} \left(f(\underbrace{0, 0, \dots, 0}_k, x_2, x_3, \dots, x_{n-k+1}) \right. \\ \left. - f(\underbrace{0, 0, \dots, 0}_k, x_1, x_2, \dots, x_{n-k}) \right). \quad (5)$$

To simplify the proof we will need the following lemma.

Lemma 2.1. If f is a number-conserving rule, then

$$f(0, 0, \dots, 0) = 0. \quad (6)$$

Write Condition (4) for a cyclic configuration of length $L \geq n$ whose all elements are equal to zero. \square

To prove that Condition (5) is necessary, consider a cyclic configuration of length $L \geq 2n - 1$ which is the concatenation of a sequence (x_1, x_2, \dots, x_n) and a sequence of $L - n$ zeros, and express that the n -input rule f is number-conserving. We obtain

$$f(0, 0, \dots, 0, x_1) + f(0, 0, \dots, 0, x_1, x_2) + \dots \\ + f(x_1, x_2, \dots, x_n) + f(x_2, x_3, \dots, x_n, 0) + \dots \\ + f(x_n, 0, \dots, 0) = x_1 + x_2 + \dots + x_n, \quad (7)$$

where all the terms of the form $f(0, 0, \dots, 0)$, which are equal to zero according to (6), have not been written. Replacing x_1 by 0 in (7) gives

$$\begin{aligned} & f(0, 0, \dots, 0, x_2) + \dots + f(0, x_2, \dots, x_n) \\ & + f(x_2, x_3, \dots, x_n, 0) + \dots + f(x_n, 0, \dots, 0) \\ & = x_2 + \dots + x_n. \end{aligned} \quad (8)$$

Subtracting (8) from (7) yields (5).

Condition (5) is obviously sufficient since, when summed on a cyclic configuration, all the left-hand side terms except the first cancel. \square

Remark 2.1. The above proof shows that if we can verify that a CA rule f is number-conserving for all cyclic configurations of length $2n - 1$, then it is number-conserving for all cyclic configurations of length $L > 2n - 1$.

The following corollaries are simple necessary conditions for a CA rule to be number-conserving.

Corollary 2.1. *If f is a one-dimensional q -state n -input number-conserving CA rule, then, for all $x \in \mathcal{Q}$,*

$$f(x, x, \dots, x) = x. \quad (9)$$

To prove (9), which is a generalization of (6), write Condition (5) for $x_1 = x_2 = \dots = x_n = x$. \square

Corollary 2.2. *If f is a one-dimensional q -state n -input number-conserving CA rule, then,*

$$\sum_{(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n} f(x_1, x_2, \dots, x_n) = \frac{1}{2}(q-1)q^n. \quad (10)$$

When we the sum (5) over $(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n$, all the left-hand side terms except the first cancel, and the sum over the remaining term is equal to $(0 + 1 + 2 + \dots + (q-1))q^{n-1} = \frac{1}{2}(q-1)q^n$. \square

It is possible to give an interesting alternative proof of Relation (10). Consider a De Bruijn cycle of length q^n . Such a cycle contains all the different n -tuples $(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n$. The existence of De Bruijn cycles is related to the existence of Eulerian circuits on a De Bruijn graph G_{n-1} . Such a graph has q^{n-1} vertices and q^n arcs. The number of arcs leaving a vertex is the same as the number which arrive. Each vertex is labeled by an $(n-1)$ -tuple over \mathcal{Q} , and the arc joining the vertex $(x_1, x_2, \dots, x_{n-1})$ to the vertex $(x_2, \dots, x_{n-1}, x_n)$ is labeled $(x_1, x_2, \dots, x_{n-1}, x_n)$. For example,

$$(0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1)$$

is a De Bruijn cycle for $q = 2$ and $n = 4$. It can be shown that there exist $q^{-n}(q!)^{q^{n-1}}$ distinct De Bruijn cycles [9]. Since all the elements of \mathcal{Q} appear an equal number of times in a De Bruijn cycle, Condition (4) implies

$$\sum_{(x_1, x_2, \dots, x_n) \in \mathcal{Q}^n} f(x_1, x_2, \dots, x_n) = (0 + 1 + 2 + \dots + (q - 1)) q^{n-1},$$

and (10) follows. \square

3. Examples of number-conserving rules

Using Condition (5) we can determine all the number-conserving CA rules for fixed values of q and n . These rules will be considered as operators governing the discrete dynamics of systems of particles whose total number is conserved. The particles occupy the cells of a one-dimensional periodic lattice subject to the condition that, at a given time, no more than $q - 1$ particles may occupy the same cell. Among these rules some have similar properties, which can be exhibited using the operators of reflection and conjugation. These two operators denoted, respectively by R and C , are defined on the set of all one-dimensional q -state n -input cellular automaton rules by

$$Rf(x_1, x_2, \dots, x_n) = f(x_n, x_{n-1}, \dots, x_1)$$

$$Cf(x_1, x_2, \dots, x_n) = q - 1 - f(q - 1 - x_1, q - 1 - x_2, \dots, q - 1 - x_n).$$

It is clear that, if f is number-conserving, then Rf , Cf , and $RCf = CRf$ have the same property. Rules f and Rf govern identical dynamics, the only difference is that, if for one rule particles flow to the right, for the other rule, they will flow in the opposite direction. To understand the difference between rules f and Cf , let us assume that a cell, which contains k particles ($1 \leq k \leq q - 1$) contains $q - 1 - k$ holes. Then, conjugation may be viewed as exchanging the roles of particles and holes. That is, if f describes a specific motion of particles then Cf describes the same rule, but for the motion of holes.

When the number of states and number of inputs are not very small, the dynamics of the particles is not clearly exhibited by the rule table of a number-conserving CA rule. A simpler and more visual picture of the rule is given by its *motion representation*. Such a motion representation may be defined as follows. List all the neighborhoods of a given occupied site represented by its site value s . Then, for each neighborhood, indicate the displacements of the s particles by a nondecreasing sequence of s integers (v_1, v_2, \dots, v_s) representing the different velocities of the s particles. Velocities are positive if particles move to the right and negative if they move to the left. Alternatively, to the sequence (v_1, v_2, \dots, v_s), we can substitute arrow(s) joining the site where are initially located the s particles to the final positions of the particle(s). A number above the arrow indicates how many particles are moving to this final position. To simplify a bit more these representations, we only list neighborhoods for which, at least one velocity v_j ($j = 1, 2, \dots, s$) is different from zero. For example, these two forms of the motion representation of 2-state 3-input Rule 184 are

$$10 \ (1) \quad \text{and} \quad \overset{1}{\curvearrowright} 10.$$

Since, for Rule 184, particles move only to the right, there is no need to indicate the state of the left neighboring site of the particle. Many other examples of motion representations for 2-state 4- and 5-input rules are given in [7].

Remark 3.1. In many applications, as one-way road car traffic, which prohibits passing, we have to assume that the particles are distinguishable. We have, therefore, to label them using an increasing sequence of integers, in order to be able to follow the motion of each individual particles. Instead of describing formally the labeling process, we will give a simple example. Suppose we have the configuration:

...	0	1	0	2	0	2	0	...
-----	---	---	---	---	---	---	---	-----

We first replace each particle in each occupied cell by the symbol \bullet

...		\bullet		$\bullet \bullet$		$\bullet \bullet$...
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and then label consecutively all the \bullet to obtain:

...		1		2 3		4 5		...
-----	--	---	--	-----	--	-----	--	-----

If now, we assume that the motion representation is

$$01 \ (-1) \quad 020 \ (-1, 1) \quad 021 \ (-1, 1) \quad 120 \ (0, 1) \quad 121 \ (0, 1)$$

$$220 \ (0, 1) \quad 221 \ (0, 1) \quad 022 \ (-1, 0) \quad 022 \ (-1, 0),$$

then, applying this rule, we obtain the new labeled configuration:

...	1		2		3 4		5	...
-----	---	--	---	--	-----	--	---	-----

which corresponds to the new configuration:

...	1	0	1	0	2	0	1	...
-----	---	---	---	---	---	---	---	-----

Note that the above labeling convention assumes that particles cannot jump above each other, and, therefore, the ordering of labels remains unchanged after each iteration. We use this convention to ensure uniqueness of the motion representation for number-conserving CA rules.

3.1. Three-state two-input number-conserving rules

There are only 4 three-state two-input number-conserving rules, which are listed in Table 1.

For radii pair (r_ℓ, r_r) , we have two possible choices, either $(0, 1)$ or $(1, 0)$. In each case, we have chosen the pair simplifying most the motion representation.

rule number	base 3 representation	(r_ℓ, r_r)
19305	222111000	(0,1)
15897	210210210	(1,0)
18561	221110110	(1,0)
16641	211211100	(0,1)

Table 1. Three-state two-input minimal number-conserving rules.

Rules 19305 and 15897 coincide with the identity. The motion representations of Rules 18561 and 16641, which are obtained from one another by reflection, are, respectively,

$$\mathbf{20} \ (0,1), \quad \mathbf{21} \ (0,1) \text{ and } \mathbf{02} \ (-1,0), \quad \mathbf{12} \ (-1,0),$$

or

$$\overset{1}{\curvearrowright} 20, \quad \overset{1}{\curvearrowright} 21, \quad \text{and} \quad \overset{1}{\curvearrowright} 02, \quad \overset{1}{\curvearrowright} 12.$$

For both rules, if a site is occupied by 2 particles, one of them move to a neighboring site, except if this site is already occupied by 2 particles.

3.2. Three-state three-input number-conserving rules

There are 144 three-state three-input number-conserving rules. They can be divided into 48 equivalence classes under the dihedral group generated by the operators R and C . Each class will be represented by the rule having the smallest rule number, and which will be called the minimal rule of the class. Tables 2 and 3 list rule numbers and corresponding base-3 representations of all 48 minimal rules.

Among these minimal rules, 18 eventually emulate the identity. Their reference numbers are: 27, 28, 29, 30, 31, 32, 33, 35, 37, 39, 40, 42, 43, 44, 45, 46, 47, 48. The motion representations of the remaining 30 rules are given in Table 4. In order to have a global view of the properties of the various minimal rules, Figures 1 and 2 represent the flow diagrams of the 30 minimal rules whose motion representations are listed in Table 4. If ρ is the average particles density and v_{av} the average particles velocity, a flow diagram shows how the flow ρv_{av} varies as a function of ρ . They are common practice in car traffic theory. Concerning these flow diagrams, one point has to be stressed. For $q > 2$, that is, if more than one particle can occupy a site, it is not always possible to choose the left and right radii of a rule so as to have $v_{av} = 0$ for $\rho = 1$. This is due to the fact that, choosing $(r_\ell, r_r) = (1, 1)$, for some rules, $\rho v_{av} = -0.5$ when $\rho = 1$. Any other choice will either increase or decrease v_{av} by one unit. The flow of the rules which emulate the identity, is equal to zero for all values of ρ .

Remark 3.2. All the rules for which the flow is non-negative for all values of the average density, could be *a priori* considered as deterministic one-lane high-

reference number	rule number	base-3 representation
1	6159136430181	210210210210210210210210
2	6159523870341	210210211210210211210210100
3	6169984499133	210211211210211100210211100
4	6171146819301	210211221210211110210100110
5	6171534259461	210211222210211111210100000
6	6181994888253	210212222210212000210101000
7	6201360244413	210221211210110100210221100
8	6202522564581	210221221210110110210110110
9	6202910004741	210221222210110111210110000
10	6213370633533	210222222210111000210111000
11	6436931290101	211210100211210211211210100
12	6447391918893	211211100211211100211211100
13	6448554239061	211211110211211110211100110
14	6448941679221	211211111211211111211100000
15	6450878870973	211211200211211200100211200
16	6452428631301	211211211211211211100100100
17	6459402308013	211212111211212000211101000
18	6462889260093	211212211211212100100101100
19	6478767664173	211221100211110100211221100
20	6480317424501	211221111211110111211110000
21	6482254616253	211221200211110200100221200
22	6483804376581	211221211211110211100110100
23	6490778053293	211222111211111000211111000
24	6491940373461	211222121211111010211000010

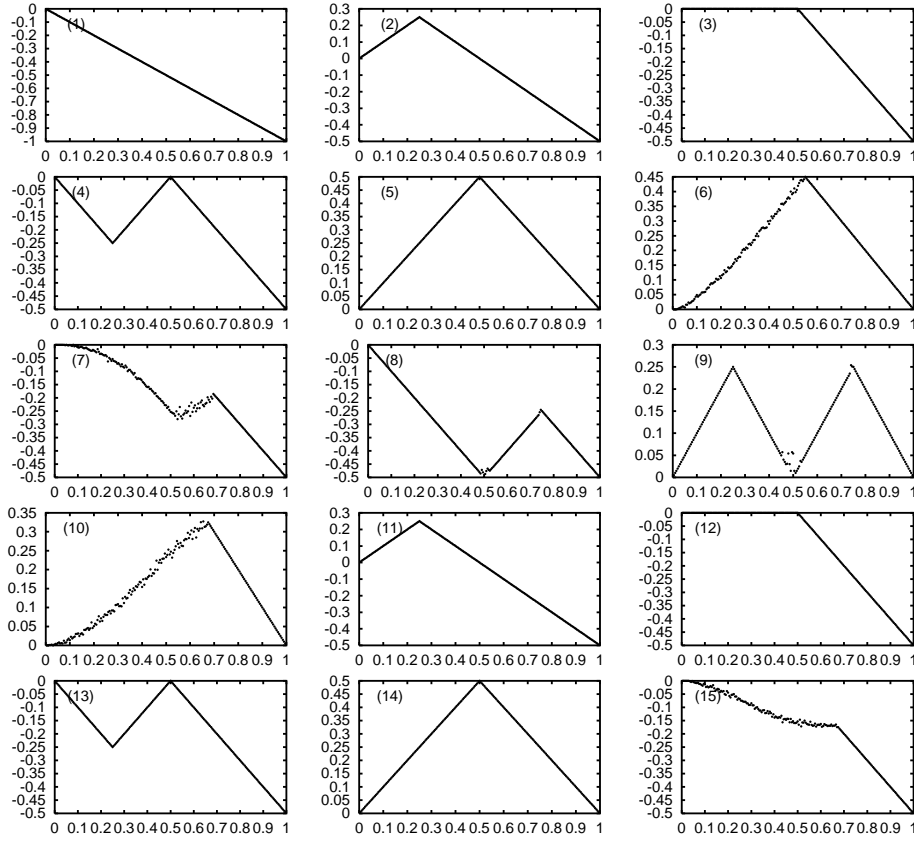
Table 2. Three-state three-input minimal number-conserving rules, Part 1.

reference number	rule number	base-3 representation
25	6494265005373	211222211211111100100111100
26	6495427325541	211222221211111110100000110
27	6726349098981	212211000212211111212100000
28	6729836051061	212211100212211211101100100
29	6736809727773	212212000212212000212101000
30	6740296679853	212212100212212100101101100
31	6757724844261	212221000212110111212110000
32	6761211796341	212221100212110211101110100
33	6768185473053	212222000212111000212111000
34	6769347793221	212222010212111010212000010
35	6771672425133	212222100212111100101111100
36	6881331565845	220100211220211100220211100
37	6893341954965	220101222220212000220101000
38	6912707311125	220110211220110100220221100
39	6924717700245	220111222220111000220111000
40	6956093445525	220121222220010000220121000
41	7158738985605	221100100221211100221211100
42	7170749374725	221101111221212000221101000
43	7174236326805	221101211221212100110101100
44	7202125120005	221111111221111000221111000
45	7205612072085	221111211221111100110111100
46	7233500865285	221121111221010000221121000
47	7448156794485	22210100022212000222101000
48	7479532539765	222111000222111000222111000

Table 3. Three-state three-input minimal number-conserving rules, Part 2.

reference number	motion representation
1	1 (-1) 2 (-1,-1)
2	10 (1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
3	20 (-1,1) 21 (-1,0) 22 (-1,0)
4	01 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
5	10 (1) 11 (1) 20 (1,1) 21 (0,1)
6	20 (1,1) 11 (1) 21 (0,1)
7	11 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
8	01 (-1) 11 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
9	10 (1) 20 (1,1) 21 (0,1)
10	20 (1,1) 21 (0,1)
11	10 (1) 2 (-1,0)
12	2 (-1,0)
13	01 (-1) 2 (-1,0)
14	10 (1) 11 (1) 20 (0,1) 21 (0,1)
15	02 (-1,-1) 12 (-1,0) 22 (-1,0)
16	10 (1) 11 (1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0)
17	11 (1) 20 (0,1) 21 (0,1)
18	020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 11 (1) 022 (-1,0)
19	11 (-1) 2 (-1,0)
20	10 (1) 20 (0,1) 21 (0,1)
21	11 (-1) 02 (-1,-1) 12 (-1,0) 22 (-1,0)
22	10 (1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0)
23	20 (0,1) 21 (0,1)
24	01 (-1) 20 (0,1) 21 (0,1)
25	020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0)
26	01 (-1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0) 122 (-1,0)
34	01 (-1) 21 (0,1)
36	21 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
38	11 (-1) 21 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0)
41	21 (-1) 2 (-1,0)

Table 4. Motion representations of the 30 minimal which do not emulate the identity.

Figure 1. *Flow diagrams, Part 1.*

way car traffic. Consider for instance Rule 6171534259461 (reference number 5), the second form of its motion representation is

$$\overset{1}{\curvearrowright} 10 \quad \overset{1}{\curvearrowright} 11 \quad \overset{2}{\curvearrowright} 20 \quad \overset{1}{\curvearrowright} 21.$$

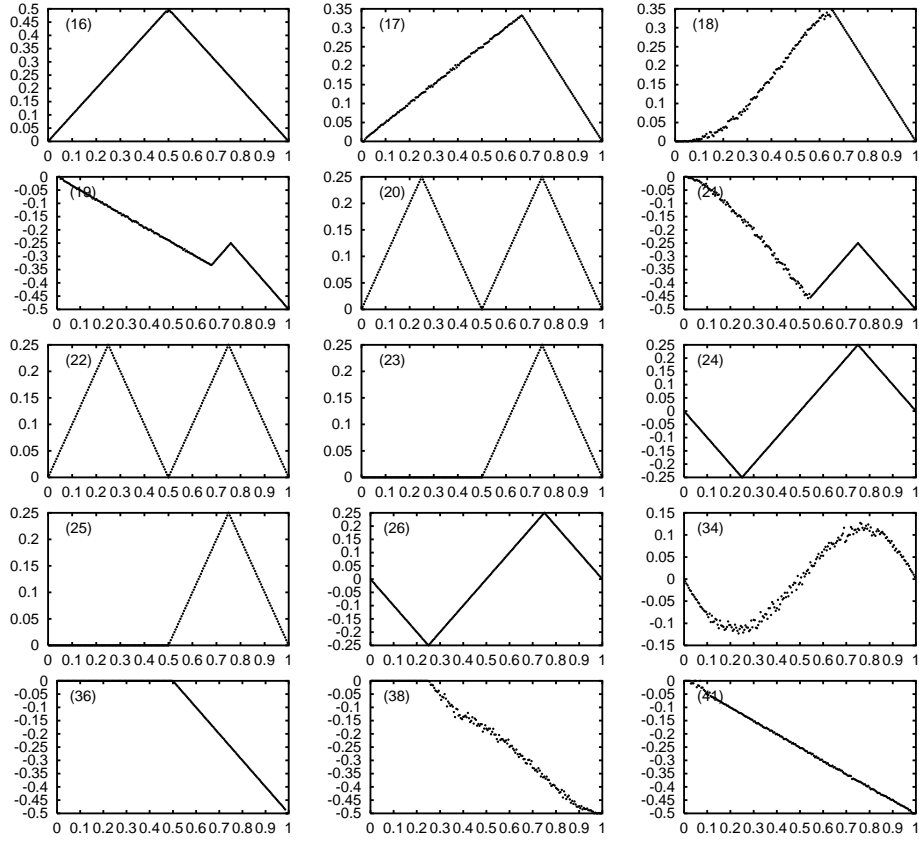
That is, as many particles as possible move from one site to the right neighboring site. This rule, which is the simplest three-state rule generalizing Rule 184, may be written

$$f(x_1, x_2, x_3) = x_2 + \min\{x_1, 2 - x_2\} - \min\{x_2, 2 - x_3\}.$$

This expression suggests a further generalization. The q -state rule defined by, for all $(x_1, x_2, x_3) \in \{0, 1, \dots, q-1\}^3$,

$$f(x_1, x_2, x_3) = x_2 + \min\{x_1, q-1-x_2\} - \min\{x_2, q-1-x_3\},$$

can be viewed as the following car traffic rule: Each cell represents a section of a one-way road between two traffic lights. The number of states q measures the maximum capacity of that section. The above evolution rule describes the way cars move from one section to the next. As for elementary CA Rule 184,

Figure 2. *Flow diagrams, Part 2.*

the flow is maximal for $\rho = 0.5$. Of course, and this goes beyond our purpose here, we could, as in the Nagel–Schreckenberg model [6], introduce some noise and say that some cars, which could move to the next section, do not do so with a probability p . The existence of traffic lights could be used to monitor traffic in order to increase the flow, and, with this in mind, we could extend to q -state rules classes of models we studied in a recent paper [10].

Remark 3.3. As mentioned above, conjugation exchanges the roles of particles and holes. Therefore, if a rule f is self-conjugate, we can always choose the rule radii (r_ℓ, r_r) such that the point $(0.5, 0)$ is a center of symmetry of its flow diagram. Rules whose reference numbers are 26 and 34 are self-conjugate and their flow diagrams have this property. In [7] we studied a few 2-state self-conjugate number-conserving rules. Some of these rules, which allow motion in both directions, govern the dynamics of ensembles of one-dimensional pseudo-random walkers. Starting from a random initial configuration, whose density ρ is exactly equal to 0.5, we followed, in the limit set, which is reached after a maximum of $L/2$ iterations, the position of a specific particle as a function of time on a ring of length $L = 5000$ for 5000 time steps. The walks are

represented in Figures 3 and 4. Since L is finite, the random walks are periodic in time. The first walk has a period equal to L , while the period of the second one is $2L$. Note that these walks are deterministic, their random character comes from the randomness of the initial configuration.

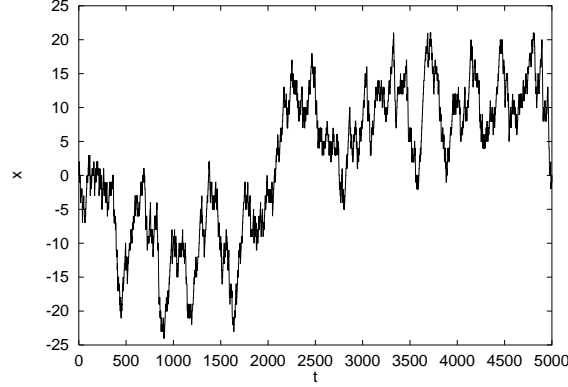


Figure 3. *Random walker evolving according to Rule 6495427325541 (no 26).*

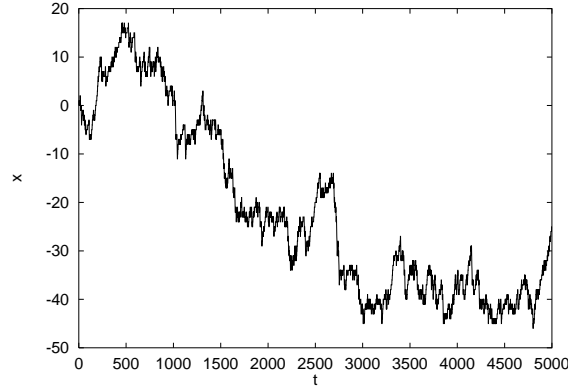


Figure 4. *Random walker evolving according to Rule 6769347793221 (no 34).*

4. Conclusion

We have established a necessary and sufficient condition for a q -state n -input CA rule to be number-conserving. For given values of q and n , this result allows to find all the rules possessing this property. As an example, we have determined all the three-state three-input number-conserving CA rules. We have listed their motion representation and studied their flow diagrams. All the rules for which the flow is non-negative for all values of the average density, can be considered as deterministic one-way road car traffic rules in which the

cells represent road sections whose car capacity is equal to $q - 1$. Among the 148 three-state three-input rules there exist two nontrivial self-conjugate rules which mimic the dynamics of an ensemble of random walkers.

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